Published by Institute of Physics Publishing for SISSA

Received: *October 12, 2006*
Revised: *December 7, 2006*
Accepted: *December 8, 2006*
Published: *December 20, 2006*

Non-perturbative formulation of time-dependent string solutions

Jean Alexandre,[a] John Ellis[ab] and Nikolaos E. Mavromatos[a]

[a] *Department of Physics, King's College London*
London WC2R 2LS, England
[b] *Theory Division, Physics Department, CERN*
CH-1211 Geneva 23, Switzerland
E-mail: jean.alexandre@kcl.ac.uk, John.Ellis@cern.ch, Nikolaos.Mavromatos@cern.ch

Abstract: We formulate here a new world-sheet renormalization-group technique for the bosonic string, which is non-perturbative in the Regge slope α' and based on a functional method for controlling the quantum fluctuations, whose magnitudes are scaled by the value of α'. Using this technique we exhibit, in addition to the well-known linear-dilaton cosmology, a new, non-perturbative time-dependent background solution. Using the freedom to perform field redefinitions in string effective actions that leave the string S-matrix invariant, we demonstrate that this solution is conformally invariant to $\mathcal{O}(\alpha')$, and we conjecture on the basis of a heuristic inductive argument that conformal invariance can be maintained to all orders in α'. This new time-dependent string solution may be applicable to primordial cosmology or to the exit from linear-dilaton cosmology at large times.

Keywords: Field Theories in Lower Dimensions, Bosonic Strings, Nonperturbative Effects.

Contents

1. **Introduction** 1

2. **Controlling the amplitudes of quantum fluctuations** 3

3. **Conformal properties of the new time-dependent solution** 5

4. **Wilsonian interpretation of the new time-dependent solution** 10

5. **Cosmological properties of the new time-dependent solution** 12

6. **Conclusions** 13

A. **A novel non-perturbative world-sheet functional renormalization method for the bosonic string** 14

B. **Geometrical properties of the graviton and dilaton backgrounds** 16

1. Introduction

The understanding of possible time-dependent string backgrounds is crucial for the development of string cosmology. A first step in this direction was the solution with a dilaton depending linearly on time that was explored in [12]. In this paper we explore further such time-dependent configurations by developing a non-perturbative renormalization-group technique developed originally for scalar field theories [2] and applied subsequently to other models [3 − 5]. We recover the linear-dilaton string solution in this approach, and also discover another possible solution which has a singularity in time. This new construction is essentially non-perturbative, so our derivation has heuristic elements. However, we are able to support it using inductive arguments.

Our approach originates from the alternative to Wilsonian renormalization flows that was proposed for a scalar model in [2], where the amplitudes of quantum fluctuations are controlled by the bare mass of the field. As this mass goes to infinity, quantum fluctuations are frozen and the system becomes classical. On the other hand, as the bare mass decreases the quantum fluctuations grow, and the system becomes dressed. The resulting evolution equation for the quantum theory is exact, and provides a resummation to all orders in $\hbar$. At the one-loop level, it recovers the well-known perturbative results but, beyond one loop, the gradient expansion used in this approach differs from the loop expansion. The same approach has subsequently been applied successfully to QED [3], to different (2+1)-dimensional models [4] and to a Yukawa theory in five dimensions [5].

We propose here a similar functional method, in which the amplitudes of quantum fluctuations are controlled by the amplitude of α' and hence the string mass scale, instead of a bare mass which is not present in the world-sheet action. This enables us to avoid introducing a non-physical running cut-off on the world sheet, and yields an evolution equation for the dilaton which is exact and provides a non-perturbative resummation to all orders in α'. We still need a world-sheet cut-off for our evolution equation, but this is fixed and can be absorbed by a rescaling of the target space coordinates. The new time-dependent string solution that we find using this approach satisfies conformal-invariance conditions that we solve explicitly at the first order in α', but we give an inductive argument that they are satisfied to all orders in α'. The new solution has a power-law singularity in the metric scale factor and a logarithmic singularity in the dilaton.

We start in section 2 by deriving the exact evolution equation for the quantum dilaton with the amplitude of α', and look for exactly marginal solutions, i.e., α'-independent configurations. We check that one such fixed-point configuration is the well-known linear dilaton/flat metric configuration [12], which was to be expected, since this configuration does not generate quantum fluctuations. We also discover a family of non-trivial fixed-point configurations, for which the target space metric is conformally flat and the metric scale factor is proportional to the second derivative of the dilaton with respect to X^0.

We examine the Weyl invariance conditions in section 3 where we show that, although the beta functions cannot vanish order by order, there exists one specific configuration in the family of fixed points for which the different beta functions display homogeneous dependences on X^0. This feature suggests that this configuration has a non-perturbative nature. If the metric of this specific fixed point is then chosen to be proportional to α', the usual expansion of the beta functions in powers of α' is not valid, and all the terms in the α' expansion have to be taken into account. We then demonstrate, using a redefinition of the dilaton and graviton fields, that it is always possible to cancel the beta functions corresponding to Weyl invariance, so that this solution is conformally invariant. We exhibit this property explicitly at one-loop order, and give a heuristic inductive argument for its validity to all orders. We also compare our new non-perturbative result with another existing in the literature [6], where mixing of σ-model-loop orders in the β functions also appears, but in a different theoretical approach.

Section 4 makes the connection with Wilsonian flows, using an infinitesimal renormalization-group technique. We show that the fixed-point configuration is an infrared-stable fixed point with regard to Wilsonian renormalization flows.

We examine the cosmological properties of this string configuration in section 5, exhibiting the singularities in the metric scale factor and the dilaton field. We also compare in that section our new non-perturbative cosmological solution with a similar (but perturbative) solution in the literature [7]. Some technical aspects of our approach are outlined in two appendices.

2. Controlling the amplitudes of quantum fluctuations

We consider a spherical world sheet with a curvature scalar $R^{(2)}$. The bare action of the σ model for the bosonic string in graviton and dilaton backgrounds reads

$$S = \frac{1}{4\pi}\int d^2\xi\sqrt{\gamma}\left\{\lambda\gamma^{ab}\eta_{\mu\nu}\partial_a X^\mu\partial_b X^\nu + R^{(2)}\phi_B(X^0)\right\}, \tag{2.1}$$

where $\eta_{\mu\nu}$ is the flat Minkowski target-space metric. Motivated by cosmology, we assume that the bare dilaton field ϕ_B is a function of the time coordinate only. The parameter λ interpolates for $1/\alpha^{'}$ and can be used to control the magnitude of the quantum fluctuations, which are proportional to $\alpha^{'}$. It parametrizes the quantum theory described by the effective action, i.e., the proper-graph-generating functional Γ_λ, which is defined in appendix A. The range of values for λ is $[1/\alpha^{'};\infty[$:

- $\lambda \to \infty$ corresponds to $\alpha^{'} \to 0$ and therefore to a classical theory: the kinetic term dominates over the bare dilaton ϕ_B term, and the theory is free;
- $\lambda \to 1/\alpha^{'}$ generates to the full quantum theory: the interactions arising from ϕ_B are gradually switched on as λ decreases from ∞ to $1/\alpha^{'}$.

We seek in this paper a λ-independent configuration of the bosonic string, which therefore, by definition, is non-perturbative since it is independent of the strength of $\alpha^{'}$.

We derive in appendix A the exact evolution equation for the proper-graph-generating functional Γ_λ with λ, which reads

$$\begin{aligned}\dot{\Gamma}_\lambda = &\frac{1}{4\pi}\int d^2\xi\sqrt{\gamma}\gamma^{ab}\eta_{\mu\nu}\partial_a X^\mu\partial_b X^\nu \\ &+\frac{\eta_{\mu\nu}}{4\pi}\mathrm{Tr}\left\{\gamma^{ab}\frac{\partial}{\partial\xi^a}\frac{\partial}{\partial\zeta^b}\left(\frac{\delta^2\Gamma_\lambda}{\delta X^\mu(\zeta)\delta X^\nu(\xi)}\right)^{-1}\right\},\end{aligned} \tag{2.2}$$

where a dot over a letter denotes a derivative with respect to λ. In eq.(2.2), the symbol of the trace (defined in eq.(A.10) of appendix A) contains the quantum corrections to S. In order to obtain physical information on the system, eq.(2.2) should in principle be integrated from $\lambda = \infty$ to $\lambda = 1/\alpha^{'}$, which is the appropriate regime of the full quantum theory. An important remark is in order, so as to emphasize the difference from Wilsonian flow and to avoid confusion: the bare kinetic term *does not* play the role of a regulator. We need a regulator (to be provided by a fixed cut-off Λ in this case) so as to define our flow with λ. The latter is therefore well defined, for a *fixed* regulator Λ.

We now derive the evolution equation for the evolution of the quantum dilaton with λ. For this, one must have knowledge of the functional dependence of Γ_λ on the quantum fields. This can be achieved by means of a gradient-expansion approximation, which assumes that, for any value of λ, Γ takes the form:

$$\begin{aligned}\Gamma_\lambda = \frac{1}{4\pi}\int d^2\xi\sqrt{\gamma}&\left\{\gamma^{ab}\kappa_\lambda(X^0)\partial_a X^0\partial_b X^0\right. \\ &\left.+\gamma^{ab}\tau_\lambda(X^0)\partial_a X^j\partial_b X^j + R^{(2)}\phi_\lambda(X^0)\right\},\end{aligned} \tag{2.3}$$

where $\kappa_\lambda, \tau_\lambda$ are λ-dependent functions of X^0 which are different for the time (X^0) and space (X^j) coordinates, since the respective quantum fluctuations are different.

We consider the limit where the radius of the world sheet goes to infinity, but keeping the curvature scalar $R^{(2)}$ finite. We show then in appendix A that the Ansatz (2.3), when plugged into the exact evolution equation (2.2), leads to:

$$\dot{\phi} = -\frac{\Lambda^2}{2R^{(2)}}\left(\frac{1}{\kappa}+\frac{D-1}{\tau}\right)+\frac{\phi''}{4\kappa^2}\ln\left(1+\frac{2\Lambda^2\kappa}{R^{(2)}\phi''}\right), \tag{2.4}$$

where Λ is the world-sheet ultraviolet cut-off, and a prime denotes a derivative with respect to X^0. After the redefinition $X^j \to \sqrt{D-1}X^j$ of the space coordinates of the string, one can see that the evolution equation (2.4) is satisfied by the well-known flat metric/linear dilaton configuration [12]:

$$\kappa = 1 = -\tau, \qquad \phi(X^0) = QX^0, \tag{2.5}$$

which shows that the latter solution is exactly marginal with respect to the flows in λ.

We now show that the evolution equation (2.4) has, besides the known configuration (2.5), another solution that is also exactly marginal with respect to the λ flow. We consider a configuration with $\kappa(X^0) = F\phi''(X^0)$, where F is a constant. For such a configuration, the evolution equation (2.4) reads

$$\kappa\dot{\phi} = -\frac{\Lambda^2}{2R^{(2)}}\left(1+(D-1)\frac{\kappa}{\tau}\right)+\frac{1}{4F}\ln\left(1+\frac{2\Lambda^2 F}{R^{(2)}}\right), \tag{2.6}$$

and one can see that it is possible to have a λ-independent solution: $\dot{\phi}=0$, if

$$\frac{\kappa}{\tau} = -\frac{1}{D-1}+\frac{R^{(2)}}{2(D-1)\Lambda^2 F}\ln\left(1+\frac{2\Lambda^2 F}{R^{(2)}}\right) = -c^2, \tag{2.7}$$

where we have taken the negative sign corresponding to Minkowski signature, as is appropriate for large cut-off Λ, in which case the ratio κ/τ is necessarily negative, and c is a positive constant. After the redefinition $X^j \to cX^j$ of the space coordinates of the string, the condition (2.7) shows that the target-space metric is conformally flat, and the non-trivial λ-independent solution of eq.(2.4) is such that

$$g_{\mu\nu}(X^0) \propto \phi''(X^0)\eta_{\mu\nu}. \tag{2.8}$$

In the stringy σ-model framework, any equilibrium solution, i.e., one that satisfies the equations of motion of a target-space effective action, must also satisfy the conformal-invariance conditions [1] on the world-sheet. *A priori*, it is not clear that the configuration (2.8),(2.7) satisfies these conditions. However, in the next section we display a more precise functional dependence for ϕ, which we conjecture satisfies the the Weyl invariance conditions. This configuration has the form:

$$ds^2 = \frac{\alpha' A}{(X^0)^2}\left(-(dX^0)^2+dX_i dX_i\right), \quad \phi = \phi_0 \ln X^0 \tag{2.9}$$

where ϕ_0 and A are to be determined.

[1] Strictly, the Weyl-invariance conditions, that take into account target-space diffeomorphisms [8].

This is still a conjecture, since the new fixed-point configuration (2.9) of the λ- low is non perturbative in α', so the corresponding Weyl anomaly coefficients are not known in a closed form. We base our conjecture that it is indeed conformally invariant on a heuristic inductive argument that there exists in principle a world-sheet renormalization scheme, reached from the standard σ-model scheme [8] by certain field redefinitions, in which the Weyl anomaly coefficients vanish. We demonstrate this explicitly to order α' and then use inductive arguments to argue that this is true to all orders in α'.

3. Conformal properties of the new time-dependent solution

To first order in $\alpha^{'}$, the beta functions for the bosonic world-sheet σ-model theory in graviton and dilaton backgrounds are [8]:

$$
\begin{aligned}
\beta^g_{\mu\nu} &= R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\phi + \frac{\alpha^{'}}{2}R_{\mu\lambda\rho\sigma}R_\nu{}^{\lambda\rho\sigma} + \mathcal{O}(\alpha^{'})^2, \\
\beta^\phi &= \frac{D-26}{6\alpha^{'}} - \frac{1}{2}\nabla^2\phi + \partial^\rho\phi\partial_\rho\phi + \frac{\alpha^{'}}{16}R_{\mu\rho\nu\sigma}R^{\mu\rho\nu\sigma} + \mathcal{O}(\alpha^{'})^2.
\end{aligned}
\tag{3.1}
$$

We consider first the tree-level beta functions for a configuration satisfying the condition $\kappa(X^0) \propto \phi^{''}(X^0)$, with the power-law dependence

$$
\begin{aligned}
\phi^{'}(X^0) &= \phi_0(X^0)^n, \\
\kappa(X^0) &= \kappa_0(X^0)^{n-1}.
\end{aligned}
\tag{3.2}
$$

In this case, we obtain (see appendix B for details):

$$
\begin{aligned}
\beta^g_{00} &= \frac{D-1}{2}\frac{n-1}{(X^0)^2} + (n+1)\phi_0(X^0)^{n-1} + \mathcal{O}(\alpha^{'}), \\
\beta^g_{jk} &= \delta_{jk}\left(\frac{(n-1)(D-2)-2}{4(X^0)^2} - \phi_0(X^0)^{n-1}\right) + \mathcal{O}(\alpha^{'}), \\
\beta^\phi &= \frac{D-26}{6\alpha^{'}} - \frac{\phi_0}{4\kappa_0}\Big(n+1+(n-1)(D-1)\Big) + \frac{\phi_0^2}{\kappa_0}(X^0)^{n+1} + \mathcal{O}(\alpha^{'}).
\end{aligned}
\tag{3.3}
$$

We observe that, for $n = -1$, each beta function is homogeneous, and we have

$$
\begin{aligned}
\beta^g_{00} &= -\frac{D-1}{(X^0)^2} + \mathcal{O}(\alpha^{'}), \\
\beta^g_{jk} &= \delta_{jk}\frac{D-1+2\phi_0}{(X^0)^2} + \mathcal{O}(\alpha^{'}), \\
\beta^\phi &= \frac{D-26}{6\alpha^{'}} + \phi_0\frac{D-1+2\phi_0}{2\kappa_0} + \mathcal{O}(\alpha^{'}).
\end{aligned}
\tag{3.4}
$$

We see that it is not possible for β^g_{00} to vanish at the tree level.

An important remark is in order, though: for the specific choice $n = -1$, the next order of each beta function is homogeneous with the tree-level term (see appendix B for details):

$$
\begin{aligned}
R_{0\mu\nu\rho} R_0^{\ \ \mu\nu\rho} &= \frac{3(D-1)}{\kappa_0 (X^0)^2}, \\
R_{j\mu\nu\rho} R_k^{\ \ \mu\nu\rho} &= -\delta_{jk} \frac{2D-1}{\kappa_0 (X^0)^2}, \\
R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} &= \frac{2(D^2-1)}{\kappa_0^2}.
\end{aligned} \tag{3.5}
$$

This is actually valid to all orders, as can readily be seen using the fact that at each order in α' the Weyl anomaly coefficients of the dilaton and graviton backgrounds consist of terms that involve products of appropriate powers of Riemann tensors, dilaton (covariant) derivatives, and the necessary contravariant metric tensors $g^{\mu\nu} \propto \phi''(X^0)\eta^{\mu\nu}$ (c.f., (2.8)) for the appropriate contractions. Inspection of such terms then reveals, in a similar way to the $\mathcal{O}(\alpha')$ terms above, homogeneous $1/(X^0)^2$ behaviours for cosmological backgrounds satisfying (3.2).

We then have, for the Ansatz (3.2) with $n = -1$,

$$
\begin{aligned}
\beta^g_{00} &= \frac{1}{(X^0)^2} \sum_{m=0}^{\infty} \xi_m \left(\frac{\alpha'}{\kappa_0} \right)^m, \\
\beta^g_{jk} &= \frac{\delta_{jk}}{(X^0)^2} \sum_{m=0}^{\infty} \zeta_m \left(\frac{\alpha'}{\kappa_0} \right)^m, \\
\beta^\phi &= \frac{1}{\alpha'} \sum_{m=0}^{\infty} \eta_m \left(\frac{\alpha'}{\kappa_0} \right)^m,
\end{aligned} \tag{3.6}
$$

where ξ_m, ζ_m, η_m are α'-independent coefficients. As a consequence, for κ_0 of the same order as α', the expansion of the beta functions in α' is no longer valid.

The next step is to argue that the configuration

$$
\begin{aligned}
\phi(X^0) &= \phi_0 \ln(X^0), \\
\kappa(X^0) &= \frac{\alpha' A}{(X^0)^2},
\end{aligned} \tag{3.7}
$$

where we write $\kappa_0 = \alpha' A$, may satisfy conformal invariance at a non-perturbative level. We exploit the fact, well-known in string theory, that at higher orders in α' the beta functions are not fixed uniquely, but can be changed by making local field redefinitions [8]: $g_{\mu\nu} \to \tilde{g}_{\mu\nu}$ and $\phi \to \tilde{\phi}$, which leave the (perturbative) string S-matrix amplitudes invariant. This possibility of field redefinition enables us to maintain conformal invariance to all orders in α'.

We illustrate this possibility with an explicit calculation to first order in α'. In our case, since we keep the target-space metric conformally flat, the redefinition of the metric must be such that $\tilde{g}_{\mu\nu}$ is proportional to $g_{\mu\nu}$, and thus we can consider the following redefinitions:

$$
\begin{aligned}
\tilde{g}_{\mu\nu} &= g_{\mu\nu} + \alpha' g_{\mu\nu} \left(b_1 R + b_2 \partial^\rho \phi \partial_\rho \phi + b_3 \nabla^2 \phi \right), \\
\tilde{\phi} &= \phi + \alpha' \left(c_1 R + c_2 \partial^\rho \phi \partial_\rho \phi + c_3 \nabla^2 \phi \right),
\end{aligned} \tag{3.8}
$$

JHEP12(2006)071

where $b_1, b_2, b_3, c_1, c_2, c_3$ are constants and R, ∇ corresponds to the metric $g_{\mu\nu}$. In the case of the configuration (C.1), we have (see appendix B for details):

$$\begin{aligned} \tilde{g}_{\mu\nu} &= g_{\mu\nu} + \frac{g_{\mu\nu}}{A}\Big(- b_1(D-1)^2 + b_2\phi_0^2 - b_3(D-1)\phi_0\Big) = (1+B)g_{\mu\nu}, \\ \tilde{\phi} &= \phi + \frac{1}{A}\Big(- c_1(D-1)^2 + c_2\phi_0^2 - c_3(D-1)\phi_0\Big) = \phi + C, \end{aligned} \tag{3.9}$$

where B, C are constants linear in $b_1, b_2, b_3, c_1, c_2, c_3$. Therefore, the redefinitions (3.8) consist of adding a constant to the dilaton and rescaling the metric, thus not changing the functional dependence of the configuration (C.1).

We now examine the changes in the beta functions after the field redefinitions (3.8). The new beta functions $\beta^g_{\mu\nu} \to \tilde{\beta}^g_{\mu\nu}$ and $\beta^\phi \to \tilde{\beta}^\phi$ are obtained via the appropriate Lie derivatives in theory space [8], as appropriate to the vector nature of the β^i functions in this space:

$$\begin{aligned} \tilde{\beta}^g_{\mu\nu} - \beta^g_{\mu\nu} &= \int (\tilde{g}_{\rho\sigma} - g_{\rho\sigma})\frac{\delta\beta^g_{\mu\nu}}{\delta g_{\rho\sigma}} + \int (\tilde{\phi} - \phi)\frac{\delta\beta^g_{\mu\nu}}{\delta\phi} \\ &\quad - \int \beta^g_{\rho\sigma}\frac{\delta(\tilde{g}_{\mu\nu} - g_{\mu\nu})}{\delta g_{\rho\sigma}} - \int \beta^\phi\frac{\delta(\tilde{g}_{\mu\nu} - g_{\mu\nu})}{\delta\phi}, \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \tilde{\beta}^\phi - \beta^\phi &= \int (\tilde{g}_{\rho\sigma} - g_{\rho\sigma})\frac{\delta\beta^\phi}{\delta g_{\rho\sigma}} + \int (\tilde{\phi} - \phi)\frac{\delta\beta^\phi}{\delta\phi} \\ &\quad - \int \beta^g_{\rho\sigma}\frac{\delta(\tilde{\phi} - \phi)}{\delta g_{\rho\sigma}} - \int \beta^\phi\frac{\delta(\tilde{\phi} - \phi)}{\delta\phi}. \end{aligned} \tag{3.11}$$

After long but straightforward computations, we obtain

$$\begin{aligned} \tilde{\beta}^g_{00} &= \frac{1}{(X^0)^2}\Bigg\{1 - D + 2C + 3BD\left(1 - D - \phi_0 + \frac{5(D-1)}{A}\right) \\ &\quad - \frac{D-1}{A}\left(\frac{D+2}{A} - D - 2\phi_0\right)\left(b_1(D-1)(6D-13) + \frac{3}{2}b_3\phi_0(D-2)\right) \\ &\quad - \left(\frac{D-26}{6} + \phi_0\frac{D-1+2\phi_0}{2A} + \frac{D^2-1}{8A^2}\right)\Big(2b_2 + (2-D)b_3\Big)\Bigg\}, \\ \tilde{\beta}^g_{jk} &= \frac{\delta_{jk}}{(X^0)^2}\Bigg\{D - 1 + 2\phi_0 + 2C + BD\left(15 - 6D - 3\phi_0 + \frac{14D-43}{A}\right) \\ &\quad - \frac{D-1}{A}\left(2\phi_0 + D - \frac{D+2}{A}\right)\left(b_1(D-1)(6D-13) + \frac{3}{2}b_3\phi_0(D-2)\right) \\ &\quad + \left(\frac{D-26}{6} + \phi_0\frac{D-1+2\phi_0}{2A} + \frac{D^2-1}{8A^2}\right)\Big(2b_2 + (2-D)b_3\Big)\Bigg\}, \\ \tilde{\beta}^\phi &= \frac{1}{\alpha'}\Bigg\{\left(\frac{D-26}{6} + \phi_0\frac{D-1+2\phi_0}{2A} + \frac{D^2-1}{8A^2}\right)\left(1 - \frac{c_3 D - 2c_2\phi_0}{A}\right) \end{aligned} \tag{3.12}$$

JHEP12(2006)071

$$+\frac{BD}{A}\left(\frac{\phi_0}{4}(3D-4)-\phi_0^2+\frac{4}{A}(D-1)(D+10)\right)-\frac{C}{A}\left(\frac{D}{2}+2\phi_0\right)$$
$$-\frac{D-1}{A^2}\left(\frac{D+2}{A}-D-2\phi_0\right)\left(3c_1D(D-1)-c_2\phi_0^2+\frac{c_3}{2}(3D-4)\phi_0\right)\Bigg\}.$$

We observe that, for any dimension D and any dilaton amplitude ϕ_0, one can always find a set of parameters $b_1, b_2, b_3, c_1, c_2, c_3$ such that all the three beta functions $\tilde{\beta}^g_{00}, \tilde{\beta}^g_{jk}, \tilde{\beta}^\phi$ vanish, since the latter are homogeneous in X^0.

In order to check that there are indeed sets of these redefinition parameters which cancel the the beta functions at first order in α', we consider the special case where $b_1 = b_3 = c_2 = 0$, so that we are left with a linear system of three equations to solve for the three variables b_2, c_1, c_3. The determinant of this system is

$$\begin{aligned}\det(A,D,\phi_0) = -\frac{(D-1)^2}{72A^6}\Big(&-24D^2A\phi_0-336DA\phi_0^2+84D^2\phi_0+36D^3\phi_0\\ &-24D\phi_0-36D^3A\phi_0+216D^2A\phi_0^2+60DA\phi_0+96A\phi_0\\ &-96\phi_0-4D^2A^2+104DA^2-3D^3+3D\Big)\\ &\times\Big(-72DA\phi_0^2+18D^2A\phi_0^2+6D^2\phi_0^2+168D\phi_0^2\\ &-4DA^2+104A^2-12DA\phi_0+12A\phi_0-24A\phi_0^2-3D^2+3\Big). \quad (3.13)\end{aligned}$$

This does not vanish in the general case, so the set of three simultaneous equations is not degenerate, and we can indeed find values of the redefinition parameters b_2, c_1, c_3 that satisfy conformal invariance. This is valid for any dimension D and dilaton amplitude ϕ_0, if the metric amplitude A is chosen in a way such that the determinant does not vanish. The determinant is a quartic function of A, so that there are at most four discrete values of A which are not allowed in this specific field redefinition, for given values of D, ϕ_0. However, these four values do not have any physical significance *a priori*, and any other choice of field redefinition, e.g., $b_2 = c_1 = c_3 = 0$, would correspond to different 'forbidden' values for A. As discussed in section 5, the specific values $D = 4, \phi_0 = -1$ are particularly interesting, since they lead to a four-dimensional Minkowski target space-time. In this case, the determinant is

$$\det(A,4,-1) = \frac{1}{2A^6}\left(909-1164A-58A^2\right)\left(723+12A+88A^2\right), \quad (3.14)$$

which vanishes for only two values of A. We emphasize again that these particular values have no physical relevance.

This analysis took into account only the first non-trivial order in α', whereas all the higher orders should also be taken into account. However, this first-order analysis provides the basis for an inductive argument. If conformal invariance is satisfied at order n in α', as in the first-order case worked out above, there are always enough parameters in the redefinitions of the metric and dilaton at the next order, leaving the string configuration unchanged, which enable the beta functions to vanish and hence conformal invariance to be satisfied at the next order $n+1$ in α'.

JHEP12(2006)071

We stress again that these arguments are only heuristic at this stage, since the β-functions and Weyl anomaly coefficients are not exactly known to all orders in α', and hence the world-sheet renormalization scheme in which they vanish is abstract. We mention at this point that it is known from standard analyses of σ models [8] that there is a formal scheme in which the dilaton dependence is simply that of one σ-model loop. For the graviton Weyl-anomaly coefficient this means that the dilaton dependence has the form of a target-space diffeomorphism : $\nabla_\mu \partial_\nu \phi$. This is *not* the scheme we use in this work, and thus the reader should bear in mind that the background (2.9) proposed here does not correspond to a standard conformal field theory that is perturbative in α'. This reflects the non-perturbative nature of the novel renormalization-group method employed in our approach.

It is instructive to compare and contrast the approach used here with the ϵ expansion used in condensed-matter problems. There, as here, the β function contains a non-vanishing term at the tree level, given in that case by dimensional analysis and here by the $\mathcal{O}((\alpha^{'})^0)$ term. There, as here, this term is cancelled by the one-loop term to yield a non-trivial infrared fixed point. In our case, this cancellation is made possible by the field-redefinition ambiguities of the string effective action, that leave the string S-matrix amplitudes invariant, and should hold to all orders in $\alpha^{'}$. In the ϵ expansion, dimensional regularization is used and the non-trivial fixed-point is modified perturbatively in higher loop orders. These provide corrections that are higher order in ϵ, notionally a small parameter whose physical value is usually $\epsilon = 1$ or 2. Nevertheless, the ϵ expansion gives useful quantitative results. In our case, there is no small expansion parameter, and the relevant parameter is the target-space scalar curvature, $\alpha' R \propto 1/A$, for the configuration (2.9), which may be large in general.

The exact value of A cannot be found in our approach, since this would require knowing the Weyl anomaly coefficients to all orders in $\alpha^{'}$. Nevertheless, an estimate for the order of magnitude of A may be found if we consider the first order in $\alpha^{'}$, for which we managed to demonstrate explicitly that the configuration (2.9) satisfies conformal invariance. At this level, we have from the dilaton β-function:

$$\beta^\phi = \frac{D-26}{6\alpha^{'}} + \frac{\phi_0}{2\kappa_0}(D-1+2\phi_0) + \frac{\alpha^{'}}{16}\frac{2(D^2-1)}{\kappa_0^2} + \mathcal{O}(\alpha^{'})^2, \tag{3.15}$$

which, with the notation $\kappa_0 = \alpha^{'} A$, leads to

$$\alpha^{'}\beta^\phi = \frac{D-26}{6} + \frac{\phi_0}{2A}(D-1+2\phi_0) + \frac{D^2-1}{8A^2} + \cdots, \tag{3.16}$$

where the dots stand for higher orders in $\alpha^{'}$. If we require β^ϕ to vanish at this order, we find for A the two possibilities

$$A = \frac{3}{D-26}\left(-\frac{\phi_0}{2}(D-1+2\phi_0) \pm \sqrt{\frac{\phi_0^2}{4}(D-1+2\phi_0)^2 - \frac{D^2-1}{2}\frac{D-26}{6}}\right). \tag{3.17}$$

To see if this result is consistent, we can take the specific case $D = 4$ and $\phi_0 = -1$, which is of cosmological relevance (cf, section 5, below). In this case we find that

$$A = \frac{3}{44}\left(-1 \pm \sqrt{111}\right). \tag{3.18}$$

The consistent solution must be positive and is therefore $A = 3(\sqrt{111} - 1)/44 \simeq 0.65$, demonstrating that our expansion parameter is of order unity, in general. This is an important difference from the perturbative ϵ expansion. Nevertheless, we believe that the qualitative conclusions drawn from the above one-loop analysis in the ϵ-expansion framework should hold in higher orders, making the similarity with our case apparent.

In closing this section, we compare our new non-perturbative result, of obtaining terms of the same order in α' for each of the terms in a σ-model-loop expansion of the β functions, with the result in [6], where a mixing of orders in the relevant β functions also appears. That work deals with σ models in D-dimensional Anti-de-Sitter space-time backgrounds, and maintains conformal invariance to all orders in α' using the well-known method of the large-N expansion, where N is in this case the target-space dimension D. In this way, the authors achieve a resummation of all the possible graphs, leading to expressions for the β functions to all orders in α', to a given order in a $1/D$ expansion. In our case, the origin of our non-perturbative results for the β-functions is quite different. For us, the α'-independent string configuration is the result of a self-consistent equation for the effective action (2.2), in the spirit of a Schwinger-Dyson equation, which by construction contains all the higher loop corrections, and is therefore different from a large-N expansion.

4. Wilsonian interpretation of the new time-dependent solution

We now exhibit the Wilsonian properties of the configuration (C.1), using the exact renormalization method of [9]. We consider an initial bare theory defined on the world sheet of the string, with cut-off Λ, as described above. The effective theory defined by the action S_k at the scale k is derived by integrating the ultraviolet degrees of freedom from Λ to k. The idea of exact renormalization methods is to perform this integration infinitesimally, from k to $k - \delta k$, which leads to an exact evolution equation for S_k in the limit $\delta k << k$. The procedure was detailed in [9], and here we reproduce only the main steps for clarity and completeness. Note that we consider here a sharp cut-off, which is possible only if we consider the evolution of the dilaton, as explained now.

We consider a Euclidean world-sheet metric, and we assume that, for each value of the energy scale k, the Euclidean action S_k has the form

$$S_k = \frac{1}{4\pi\alpha'}\int d^2\xi\sqrt{\gamma}\Big\{\gamma^{ab}\kappa_k(X^0)\delta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu + \alpha' R^{(2)}\phi_k(X^0)\Big\}, \tag{4.1}$$

where $R^{(2)}$ is the curvature scalar of the spherical world sheet. The integration of the ultraviolet degrees of freedom is implemented in the following way. We write the dynamical fields X^μ as $X^\mu = x^\mu + y^\mu$, where the x^μ are the infrared fields with non-vanishing Fourier components for $|p| \leq k - \delta k$, and the y^μ are the degrees of freedom to be integrated out,

JHEP12(2006)071

with non-vanishing Fourier components for $k - \delta k < |p| \leq k$ only. An infinitesimal step of the renormalization group transformation reads (we take the limit of a flat world-sheet metric, keeping $R^{(2)}$ finite):

$$\begin{aligned} & \exp\left(-S_{k-\delta k}[x] + S_k[x]\right) \\ &= \exp\left(S_k[x]\right) \int \mathcal{D}[y] \exp\left(-S_k[x+y]\right) \\ &= \int \mathcal{D}[y] \exp\left(-\int_k \frac{\delta S_k[x]}{\delta y^\mu(p)} y^\mu(p) - \frac{1}{2}\int_k \int_k \frac{\delta^2 S_k[x]}{\delta y^\mu(p)\delta y^\nu(q)} y^\mu(p) y^\nu(q)\right), \\ & \qquad + \text{higher orders in } \delta k, \end{aligned} \tag{4.2}$$

where $\int_k$ represents the integration over Fourier modes for $k - \delta k < |p| \leq k$. Higher-order terms in the expansion of the action are indeed of higher order in δk, since each integral involves a new factor of δk. The only relevant terms are of first and second order in δk [9], which are at most quadratic in the dynamical variable y, and therefore lead to a Gaussian integral. We then have

$$\begin{aligned} \frac{S_k[x] - S_{k-\delta k}[x]}{\delta k} &= \frac{\mathrm{Tr}_k}{\delta k}\left\{\frac{\delta S_k[x]}{\delta y^\mu(p)}\left(\frac{\delta^2 S_k[x]}{\delta y^\mu(p)\delta y^\nu(q)}\right)^{-1}\frac{\delta S_k[x]}{\delta y^\nu(q)}\right\} \\ & \quad -\frac{\mathrm{Tr}_k}{2\delta k}\left\{\ln\left(\frac{\delta^2 S_k[x]}{\delta y^\mu(p)\delta y^\nu(q)}\right)\right\} + \mathcal{O}(\delta k), \end{aligned} \tag{4.3}$$

where the trace Tr_k is to be taken in the shell of thickness δk, and is therefore proportional to δk.

We are interested in the evolution equation of the dilaton, for which it is sufficient to consider a constant infrared configuration x^μ [2]. In this situation, the first term on the right-hand side of eq.(4.3), which is a tree-level term, does not contribute: $\delta S_k/\delta y^\mu(p)$ is proportional to $\delta^2(p)$, and thus has no overlap with the domain of integration $|p| = k$. We are therefore left with the second term, which arises from quantum fluctuations, and the limit $\delta k \to 0$ gives, with the Ansatz (4.1),

$$R^{(2)}\partial_k \phi_k(x^0) = -k \ln\left(\frac{2\kappa_k(x^0)k^2 + \alpha' R^{(2)}\phi_k''(x^0)}{2\kappa_k(1)k^2 + \alpha' R^{(2)}\phi_k''(1)}\left(\frac{\kappa_k(x^0)}{\kappa_k(1)}\right)^{D-1}\right), \tag{4.4}$$

where a prime denotes a derivative with respect to X^0 and we chose the dilaton to vanish for $x^0 = 1$. Eq.(4.4) provides a resummation in α', since the quantities in the logarithm are the running, dressed quantities. As a result, the evolution equation (4.4) is exact within the framework of the Ansatz (4.1), and is non-perturbative.

One can easily see that a linear dilaton/flat metric configuration [12]

$$\kappa(x^0) = 1 \qquad \phi(x^0) = Qx^0, \tag{4.5}$$

where the constant Q is independent of k, satisfies the evolution equation (4.4). This is an exactly marginal configuration, independent of the Wilsonian scale k, which is expected because it does not generate any quantum fluctuations.

[2] This is why a sharp cut-off can be used: the singular terms that could arise from the θ function, characterizing the sharp cut-off, are not present, since the derivatives of the infrared field vanish.

JHEP12(2006)071

We now come back to the configuration (C.1), and look for a similar exact solution of the renormalization-group equation (4.4), in which the graviton background is given by the same expression as in eq.(C.1), but the dilaton has the form

$$\phi_k(x^0) = \eta_k \ln(x^0), \tag{4.6}$$

where η_k is a function of k. It is easy to see that such a configuration indeed satisfies eq.(4.4), provided that η_k satisfies $d\eta_k/dk = 2Dk/R^{(2)}$, and hence

$$\phi_k(x^0) = \left(\phi_0 + \frac{Dk^2}{R^{(2)}}\right) \ln(x^0), \tag{4.7}$$

where ϕ_0 is the constant of integration.

Therefore, we have been able to find an exact solution of the renormalization-group equation (4.4), which tends to the solution (C.1) in the infrared limit $k \to 0$, with a vanishing derivative

$$\partial_k \phi_k(x^0) \to 0^+, \tag{4.8}$$

and thus we conclude that the configuration (C.1) is a Wilsonian infrared-stable fixed point.

5. Cosmological properties of the new time-dependent solution

We now examine the physical significance of the new non-trivial fixed-point solution (C.1), and discuss briefly its cosmological implications. This leads to a value of the constant ϕ_0 that appears in the configuration (C.1).

The relation between the physical metric in the Einstein frame and the string metric is given by [12]

$$\begin{aligned} ds^2 &= dt^2 - a^2(t) dx^k dx^k \\ &= \kappa(x^0) \exp\left\{-\frac{4\phi(x^0)}{D-2}\right\} \left(dx^0 dx^0 - dx^k dx^k\right), \end{aligned} \tag{5.1}$$

where $a(t)$ is the scale factor of a spatially flat Robertson-Walker-Friedmann Universe, and the x^μ are the zero modes of X^μ. From the configuration (C.1), we have

$$\frac{dt}{dx^0} = \varepsilon \sqrt{|\kappa_0|} (x^0)^{-1-\frac{2\phi_0}{D-2}}, \tag{5.2}$$

where $\varepsilon = \pm 1$, such that

$$t = T + \sqrt{|\kappa_0|} \frac{(D-2)}{2|\phi_0|} (x^0)^{-\frac{2\phi_0}{D-2}}, \tag{5.3}$$

where T is a constant. We find then a power law for the evolution of the scale factor:

$$a(t) = a_0 |t - T|^{\frac{D-2}{2\phi_0}+1}, \tag{5.4}$$

which is in general singular as $t \to T$.

In order to have a Minkowski target space, one needs $D-2+2\phi_0 = 0$. As was discussed in section 3, when dealing with the conformal properties of the configuration (C.1), the choices of D and ϕ_0 are free, and lead to the determination of κ_0 (in a way which has not been determined yet). As a consequence, for a given dimension D, it is always possible to choose ϕ_0 so that the target space is static and flat. It may therefore find an application to the exit phase from the linearly expanding Universe associated with the linear dilaton of [12].

We note that, in terms of the Einstein time t, the dilaton can be written, up to a constant, as:

$$\phi = -\frac{D-2}{2}\ln|t-T|. \tag{5.5}$$

We observe that, like the scale factor (5.4), the dilaton has a singularity as $t \to T$. It would be interesting to explore the applicability of this configuration to primordial cosmology. The sign of the expression (5.5) for the dilaton when $D > 2$ ensures that the string coupling is small at large times.

In closing this section, we mention for completeness that our solution (C.1) (or (5.4), (5.5) in the Einstein frame) should be compared and contrasted with the (isotropic-case) solution of [7], which describes a D-dimensional Universe whose constant-time slices are (D-1)-dimensional tori with time-dependent 'rolling" radii, whose flow is attributed to the existence of a rolling dilaton field. First, it should be remarked that the work of [7] is perturbative in α', unlike our construction. Moreover, our main point in this work has been to associate the non-trivial fixed point solution (C.1) to a marginal configuration of flow with respect to a novel control parameter or, equivalently, to an infrared fixed point of a Wilsonian flow (see section 4). In this way, we have provided arguments for the rôle of the solution, especially in the Minkowski case, as providing an exit phase from a linearly-expanding Universe. None of these aspects apply to the work of [7].

6. Conclusions

We have proposed in this paper a new non-perturbative renormalization-group technique for the bosonic string, based on a functional method for controlling the quantum fluctuations, whose magnitudes are scaled by the value of α'. Using this technique, we have exhibited a new, non-perturbative time-dependent background solution. Using the field redefinition ambiguities of the target-space effective action, which leave the string S-matrix invariant, we have demonstrated that this solution is conformally invariant to $\mathcal{O}(\alpha')$, and we have made a conjecture, based on a heuristic and inductive argument, that conformal invariance can be maintained to all orders in α'. We stress once again that our work, which is based on the flow equations of a non-perturbative Schwinger-Dyson-type effective action, is different in spirit from other work in the literature [6], where expressions for the β functions to all orders in α' are obtained in a large-N-treatments, where N is the number of target-space dimensions.

This new non-perturbative time-dependent background solution has related singularities in both the metric scale factor and the dilaton value at a specific value of the time

JHEP12(2006)071

in the Einstein frame. A full exploration of the possible cosmological applications of this solution lies beyond the scope of this paper, but we do note two interesting possibilities. One is that the temporal singularity might be relevant for primordial cosmology, i.e., the beginning of the Big Bang. The second possible application could be to describe the exit phase from the linearly expanding Universe associated with the linear dilaton of [12].

These are two phenomenological tasks for future work on this new non-perturbative time-dependent background solution. It is also desirable to explore in more detail the formal underpinnings of the solution. In particular, it is necessary to improve on our heuristic inductive argument that its conformal invariance may be maintained to all orders in $\alpha^{'}$. We also note that the non-perturbative renormalization-group technique proposed here may have applications to other aspects of string theory.

Acknowledgments

The work of J.E. and N.E.M. was supported in part by the European Union through the Marie Curie Research and Training Network UniverseNet (MRTN-CT-2006-035863).

A. A novel non-perturbative world-sheet functional renormalization method for the bosonic string

We start with the bare action for the bosonic string on a Euclidean world sheet, expressed in terms of microscopic fields $\tilde{X}^\mu$ defined in the bare theory:

$$S_B = \frac{1}{4\pi}\int d^2\xi\sqrt{\gamma}\left\{\gamma^{ab}\lambda\eta_{\mu\nu}\partial_a\tilde{X}^\mu\partial_b\tilde{X}^\nu + R^{(2)}\phi_B(\tilde{X}^0)\right\}, \tag{A.1}$$

to which we add the source term

$$S_S = \int d^2\xi\sqrt{\gamma}R^{(2)}\eta_{\mu\nu}V^\mu\tilde{X}^\nu, \tag{A.2}$$

in order to define the classical fields. The corresponding partition function Z and the generating functional of the connected graphs W are related as usual:

$$Z = \int\mathcal{D}[\tilde{X}]e^{-S_B-S_S} = e^{-W}. \tag{A.3}$$

The classical fields X^μ are defined by

$$X^\mu = \frac{1}{Z}\int\mathcal{D}[\tilde{X}]\tilde{X}^\mu e^{-S_B-S_S} = \frac{1}{Z}\left\langle\tilde{X}^\mu\right\rangle, \tag{A.4}$$

and are obtained by differentiating W with respect to the source V_μ:

$$\frac{1}{\sqrt{\gamma_\xi}R^{(2)}_\xi}\frac{\delta W}{\delta V_\mu(\xi)} = X^\mu(\xi). \tag{A.5}$$

The second derivative of W is then

$$\frac{1}{\sqrt{\gamma_\zeta\gamma_\xi}R^{(2)}_\zeta R^{(2)}_\xi}\frac{\delta^2 W}{\delta V_\mu(\zeta)\delta V_\nu(\xi)} = X^\nu(\xi)X^\mu(\zeta) - \frac{\left\langle\tilde{X}^\nu(\xi)\tilde{X}^\mu(\zeta)\right\rangle}{Z}. \tag{A.6}$$

Inverting the relation (A.5) between V_μ and X^μ, we then introduce the Legendre transform of W, namely the functional Γ responsible for the generation of proper graphs:

$$\Gamma = W - \int d^2\xi \sqrt{\gamma} R^{(2)} V^\mu X_\mu. \tag{A.7}$$

The functional derivatives of Γ are:

$$\begin{aligned} \frac{1}{\sqrt{\gamma_\xi} R^{(2)}_\xi} \frac{\delta \Gamma}{\delta X^\mu(\xi)} &= -V_\mu(\xi), \\ \frac{1}{\sqrt{\gamma_\xi \gamma_\zeta} R^{(2)}_\xi R^{(2)}_\zeta} \frac{\delta^2 \Gamma}{\delta X^\nu(\zeta) \delta X^\mu(\xi)} &= -\left(\frac{\delta^2 W}{\delta V_\nu(\zeta) \delta V_\mu(\xi)} \right)^{-1}. \end{aligned} \tag{A.8}$$

The evolution of W with λ is given by

$$\begin{aligned} \dot{W} &= \frac{1}{4\pi Z} \int d^2\xi \sqrt{\gamma} \gamma^{ab} \eta_{\mu\nu} \left\langle \partial_a \tilde{X}^\mu \partial_b \tilde{X}^\nu \right\rangle \\ &= \frac{\eta_{\mu\nu}}{4\pi Z} \mathrm{Tr} \left\{ \gamma^{ab} \frac{\partial}{\partial \xi^a} \frac{\partial}{\partial \zeta^b} \left\langle \tilde{X}^\mu(\xi) \tilde{X}^\nu(\zeta) \right\rangle \right\}, \end{aligned} \tag{A.9}$$

where a dot denotes a derivative with respect to λ, and the trace is defined by

$$\mathrm{Tr}\{\cdots\} = \int d^2\xi d^2\zeta \sqrt{\gamma_\xi \gamma_\zeta} \{\cdots\} \delta^2(\xi - \zeta). \tag{A.10}$$

We then obtain

$$\begin{aligned} \dot{W} &= \frac{1}{4\pi} \int d^2\xi \sqrt{\gamma} \gamma^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \\ &\quad - \frac{\eta_{\mu\nu}}{4\pi} \mathrm{Tr} \left\{ \gamma^{ab} \frac{\partial}{\partial \xi^a} \frac{\partial}{\partial \zeta^b} \left(\frac{1}{\sqrt{\gamma_\xi \gamma_\zeta} R^{(2)}_\xi R^{(2)}_\zeta} \frac{\delta^2 W}{\delta V_\mu(\zeta) \delta V_\nu(\xi)} \right) \right\}. \end{aligned} \tag{A.11}$$

Finally, the evolution of Γ is obtained by noting that the independent variables of Γ are X^μ and λ, and that

$$\begin{aligned} \dot{\Gamma} &= \dot{W} + \int d^2\xi \frac{\partial W}{\partial V_\mu} \dot{V}_\mu - \int d^2\xi \sqrt{\gamma} R \dot{V}_\mu X^\mu \\ &= \dot{W}. \end{aligned} \tag{A.12}$$

Using eqs.(A.8), (A.11) and (A.12), we obtain finally the following evolution equation for Γ:

$$\begin{aligned} \dot{\Gamma} &= \frac{1}{4\pi} \int d^2\xi \sqrt{\gamma} \gamma^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \\ &\quad + \frac{\eta_{\mu\nu}}{4\pi} \mathrm{Tr} \left\{ \gamma^{ab} \frac{\partial}{\partial \xi^a} \frac{\partial}{\partial \zeta^b} \left(\frac{\delta^2 \Gamma}{\delta X^\mu(\zeta) \delta X^\nu(\xi)} \right)^{-1} \right\}. \end{aligned} \tag{A.13}$$

We now assume the following functional dependence:

$$\Gamma = \frac{1}{4\pi} \int d^2\xi \sqrt{\gamma} \left\{ \gamma^{ab} \left(\kappa_\lambda(X^0) \partial_a X^0 \partial_b X^0 + \tau_\lambda(X^0) \partial_a X^k \partial_b X^k \right) + R^{(2)} \phi_\lambda(X^0) \right\}. \tag{A.14}$$

JHEP12(2006)071

With the metric $\gamma^{ab} = \delta^{ab}$ and a constant configuration $X^0(\xi) = x^0$, the second functional derivatives of Γ are

$$\begin{aligned}
\frac{\delta^2\Gamma}{\delta X^0(\zeta)\delta X^0(\xi)} &= -\frac{\kappa}{2\pi}\Delta\delta^2(\xi-\zeta) + \frac{R^{(2)}\phi''}{4\pi}\delta^2(\xi-\zeta), \\
\frac{\delta^2\Gamma}{\delta X^j(\zeta)\delta X^k(\xi)} &= -\frac{\tau}{2\pi}\Delta\delta^2(\xi-\zeta)\delta_{jk},
\end{aligned} \tag{A.15}$$

where a prime denotes a derivative with respect to x_0. The second functional derivatives of Γ read then, in Fourier components,

$$\begin{aligned}
\frac{\delta^2\Gamma}{\delta X^0(p)\delta X^0(q)} &= \frac{1}{4\pi}\left(2\kappa p^2 + R^{(2)}\phi''\right)\delta^2(p+q) \\
\frac{\delta^2\Gamma}{\delta X^j(p)\delta X^k(q)} &= \frac{\tau p^2}{2\pi}\delta^2(p+q)\delta_{jk}.
\end{aligned} \tag{A.16}$$

The area of a sphere with curvature scalar $R^{(2)}$ is $8\pi/R^{(2)}$, so with the constant configuration $X^0 = x_0$ we have

$$\Gamma = 2\phi_\lambda(x_0), \tag{A.17}$$

and the trace appearing in eq.(A.13) is

$$\begin{aligned}
\frac{1}{4\pi}\mathrm{Tr}\{\partial\partial(\delta^2\Gamma)^{-1}\} &= -\int\frac{d^2p}{(2\pi)^2}\left(\frac{p^2}{2\kappa p^2 + R^{(2)}\phi''} + \frac{D-1}{2\tau}\right)\frac{8\pi}{R^{(2)}} \\
&= -\frac{\Lambda^2}{R^{(2)}}\left(\frac{1}{\kappa} + \frac{D-1}{\tau}\right) + \frac{\phi''}{2\kappa^2}\ln\left(1 + \frac{2\Lambda^2\kappa}{R^{(2)}\phi''}\right).
\end{aligned} \tag{A.18}$$

To compute this trace, we used the fact that

$$\delta^2(p=0) = \text{ world-sheet area } = \frac{8\pi}{R^{(2)}}. \tag{A.19}$$

The evolution equation for ϕ is finally obtained by putting together results (A.13), (A.17) and (A.18):

$$\dot\phi = -\frac{\Lambda^2}{2R^{(2)}}\left(\frac{1}{\kappa} + \frac{D-1}{\tau}\right) + \frac{\phi''}{4\kappa^2}\ln\left(1 + \frac{2\Lambda^2\kappa}{R^{(2)}\phi''}\right). \tag{A.20}$$

B. Geometrical properties of the graviton and dilaton backgrounds

For the target-space metric $g_{\mu\nu}(X^0) = \kappa(X^0)\eta_{\mu\nu}$, the non-vanishing Christofel symbols are

$$\Gamma^j{}_{0k} = \delta^j_k\frac{\kappa'}{2\kappa}, \qquad \Gamma^0{}_{00} = \frac{\kappa'}{2\kappa}, \qquad \Gamma^0{}_{jk} = \delta_{jk}\frac{\kappa'}{2\kappa}, \tag{B.1}$$

so that the non vanishing covariant derivatives of the dilaton are

$$\begin{aligned}
\nabla_0\nabla_0\phi &= \phi'' - \frac{\kappa'}{2\kappa}\phi', \\
\nabla_j\nabla_k\phi &= -\delta_{jk}\frac{\kappa'}{2\kappa}\phi',
\end{aligned} \tag{B.2}$$

and the non-vanishing components of the Riemann and Ricci tensors are

$$
\begin{aligned}
R_{0j0}{}^{k} &= -\delta_j^k \left(\frac{\kappa^{'}}{2\kappa}\right)^{'}, \\
R_{lkm}{}^{j} &= \left(\frac{\kappa^{'}}{2\kappa}\right)^2 \left(\delta_{lm}\delta_k^j - \delta_{km}\delta_l^j\right), \\
R_{00} &= -(D-1)\left(\frac{\kappa^{'}}{2\kappa}\right)^{'}, \\
R_{jk} &= \delta_{jk}\left(\frac{\kappa^{'}}{2\kappa}\right)^{'} + \delta_{jk}(D-2)\left(\frac{\kappa^{'}}{2\kappa}\right)^2 .
\end{aligned}
\tag{B.3}
$$

We then have

$$
\begin{aligned}
R &= -\frac{D-1}{\kappa}\left(\frac{\kappa^{'}}{\kappa}\right)^{'} - (D-1)\frac{(D-2)}{\kappa}\left(\frac{\kappa^{'}}{\kappa}\right)^2, \\
R_{0\mu\nu\rho}R_0{}^{\mu\nu\rho} &= (D-1)\frac{3}{\kappa}\left[\left(\frac{\kappa^{'}}{2\kappa}\right)^{'}\right]^2, \\
R_{j\mu\nu\rho}R_k{}^{\mu\nu\rho} &= -\delta_{jk}\frac{3}{\kappa}\left[\left(\frac{\kappa^{'}}{2\kappa}\right)^{'}\right]^2 - 2\delta_{jk}\frac{D-2}{\kappa}\left(\frac{\kappa^{'}}{2\kappa}\right)^4, \\
R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} &= 6\frac{D-1}{\kappa^2}\left[\left(\frac{\kappa^{'}}{2\kappa}\right)^{'}\right]^2 + 2\frac{(D-1)(D-2)}{\kappa^2}\left(\frac{\kappa^{'}}{2\kappa}\right)^4 .
\end{aligned}
\tag{B.4}
$$

Since $\tilde{g}_{\rho\sigma} - g_{\rho\sigma} = B\kappa\eta_{\rho\sigma}$, the functional derivatives are taken using the relation

$$
\int (\tilde{g}_{\rho\sigma} - g_{\rho\sigma})\frac{\delta(\cdots)}{\delta g_{\rho\sigma}} = BD\int \kappa\frac{\delta(\cdots)}{\delta\kappa}. \tag{B.5}
$$

References

[1] I. Antoniadis, C. Bachas, J.R. Ellis and D.V. Nanopoulos, *Cosmological string theories and discrete inflation*, *Phys. Lett.* **B 211** (1988) 393; *An expanding universe in string theory*, *Nucl. Phys.* **B 328** (1989) 117; *Comments on cosmological string solutions*, *Phys. Lett.* **B 257** (1991) 278.

[2] J. Alexandre and J. Polonyi, *Functional Callan-Symanzik equation*, *Ann. Phys. (NY)* **288** (2001) 37 [hep-th/0010128].

[3] J. Alexandre, J. Polonyi and K. Sailer, *Functional Callan-Symanzik equation for QED*, *Phys. Lett.* **B 531** (2002) 316 [hep-th/0111152].

[4] J. Alexandre, *A control on quantum fluctuations in 2+1 dimensions*, *Phys. Rev.* **D 68** (2003) 085016 [hep-th/0306039]; *Non-renormalization for planar Wess-Zumino model*, *Phys. Lett.* **B 594** (2004) 381 [hep-th/0311266].

JHEP12(2006)071

[5] J. Alexandre and K. Farakos, *Control of quantum fluctuations for a Yukawa interaction in the Kaluza Klein picture*, *New J.~Phys.* **8** (2006) 198 [hep-ph/0603016].

[6] G. Michalogiorgakis and S.S. Gubser, *Heterotic non-linear sigma models with anti-de Sitter target spaces*, *Nucl. Phys.* **B 757** (2006) 146 [hep-th/0605102].

[7] M.T. Mueller, *Rolling radii and a time dependent dilaton*, *Nucl. Phys.* **B 337** (1990) 37.

[8] R.R. Metsaev and A.A. Tseytlin, *Order alpha-prime (two loop) equivalence of the string equations of motion and the sigma model weyl invariance conditions: dependence on the dilaton and the antisymmetric tensor*, *Nucl. Phys.* **B 293** (1987) 385.

[9] F.J. Wegner and A. Houghton, *Renormalization group equation for critical phenomena*, *Phys. Rev.* **A 8** (1973) 401.

JHEP12(2006)071

Received: *April 3, 2007*

Addendum

by authors Jean Alexandre and Nikolaos Mavromatos

Using a novel, non-perturbative, time-dependent string configuration derived in [1], we present here an argument which selects new critical dimensions for the target space time of a bosonic sigma model, with $D = 4$ the lowest non trivial value. This argument is based on the properties of the partition function after a target space Wick rotation.

In [10], the authors were interested in a time-dependent configuration of the bosonic string, relevant to the description of a spatially-flat Robertson Walker Universe, with metric $ds^2 = dt^2 - a^2(t)(d\vec{x})^2$, where t is the time in the Einstein frame, and $a(t)$ is the scale factor. It was argued that the following time-dependent configuration

$$S = \frac{1}{4\pi\alpha'} \int d^2\xi\sqrt{\gamma} \left\{ \gamma^{ab} \frac{A\eta_{\mu\nu}}{(X^0)^2} \partial_a X^\mu \partial_b X^\nu + \alpha' R^{(2)} \phi_0 \ln(X^0) \right\}, \tag{C.1}$$

where A and ϕ_0 are constants, is a fixed point of the α' flow equations derived in [10]. It was then conjectured that this configuration satisfies Weyl invariance conditions to all orders in the Regge slope α'. This result holds for any target space dimension D, and the constant A depends on ϕ_0 and D, in a way that could not be determined, since this would require knowledge of the Weyl anomaly coefficients to all orders in α'.
The corresponding scale factor was then shown to be a power law

$$a(t) \propto t^{1+\frac{D-2}{2\phi_0}}, \tag{C.2}$$

such that, if the following relation holds

$$D = 2 - 2\phi_0, \tag{C.3}$$

the target space is static and flat (Minkowski Universe).

In this note, we wish to study the properties of the configuration (C.1) under the target space Wick rotation, as appropriate for a well-defined world-sheet path integral, since in such a case the two-dimensional field $X^0(\xi)$ does not have negative norm:

$$X^0 \to iX^0. \tag{C.4}$$

For a given world sheet Euclidean metric, the partition function obtained from the action (C.1) is

$$\mathcal{Z} = \frac{\int \mathcal{D}[X^\mu] \exp(-S)}{\int \mathcal{D}[X^\mu]}, \tag{C.5}$$

where $\int \mathcal{D}[X^\mu] = V$ is the target space volume. The analytic continuation (C.4) implies the expected change of the target space Minkowski metric into a Euclidean one, since

$$\begin{aligned} \frac{\eta_{\mu\nu}}{(X^0)^2} \partial_a X^\mu \partial_b X^\nu &\to \frac{\partial_a(iX^0)\partial_b(iX^0)}{(iX^0)^2} - \delta_{jk} \frac{\partial_a X^j \partial_b X^k}{(iX^0)^2} \\ &= \frac{\delta_{\mu\nu}}{(X^0)^2} \partial_a X^\mu \partial_b X^\nu. \end{aligned} \tag{C.6}$$

We are therefore led to the following action

$$S \to \tilde{S} = S_E + i\frac{\pi}{2}\phi_0\chi, \tag{C.7}$$

where the (target space and world sheet) Euclidean action is

$$S_E = \frac{1}{4\pi\alpha'}\int d^2\xi\sqrt{\gamma}\left\{\gamma^{ab}\frac{A\delta_{\mu\nu}}{(X^0)^2}\partial_a X^\mu \partial_b X^\nu + \alpha' R^{(2)}\phi_0 \ln(X^0)\right\}, \tag{C.8}$$

and

$$\chi = \frac{1}{4\pi}\int d^2\xi\sqrt{\gamma}\gamma^{ab}R^{(2)} \tag{C.9}$$

is the Euler characteristic of the world sheet. For a closed world sheet, without cross-caps, we have [11]

$$\chi = 2 - 2g, \tag{C.10}$$

where g is the number of handles, such that the partition function becomes, after the Wick rotation (C.4),

$$\mathcal{Z} \to \tilde{\mathcal{Z}} = \mathcal{Z}_E \exp\{i\pi(1-g)\phi_0\}, \tag{C.11}$$

where $\mathcal{Z}_E$ is the partition function corresponding to the action S_E. For $\tilde{\mathcal{Z}}$ to be real, we need the quantization condition

$$(1-g)\phi_0 = n, \tag{C.12}$$

where n is an integer, different from 0, since $\phi_0 \neq 0$ in the solution of [10]. Together with the relation (C.3), this quantization of the dilaton amplitude yields for the dimension D

$$D = 2 - \frac{2n}{1-g}, \tag{C.13}$$

in order to have a Minkowski target space. If we consider the case of the spherical world sheet, with $g = 0$, and for which the configuration (C.1) was derived, the allowed dimensions are then obtained for $n < 0$ and are the even integers not smaller than 4

$$D = 4, 6, 8, 10, ..., 26, ... \tag{C.14}$$

As a consequence, $D = 4$ is the minimum dimension for which the analytic continuation (C.4) can be performed consistently in the partition function (C.5), and for which the string propagates in a Minkowski space time background.

Before closing we would like to stress the difference of our approach to that of [12, 13], as far as the target-space dimensionality is concerned. In those works, the target time X^0 [12] and the Liouville mode φ [13] have a similar rôle, namely to restore the central charge of the sigma model theory to its critical value, fixed by the ghost fields. This was due to the fact that the corresponding field theories in question were simple ones, with canonically normalized kinetic terms for the sigma model fields, where the resulting target space dimensionality depends on the value of the dilaton and/or Liouville amplitude.

Instead, in our approach [10], the fields X^μ are not canonically normalized, and the above-mentioned results do not apply. As argued in [10], the conformal invariance is satisfied for any target-space dimensionality D, independently of the dilaton amplitude, provided the target space has a Minkowski signature. However, as we have discussed here, the condition (C.3) for a static target space and the analytic continuation (C.4) to a Euclidean Universe imply the quantization (C.13) of D.

Our result is also different from the strongly-coupled Liouville string regime discussed in [14], for central charges $1 < c < 25$, where different critical values for $D = 7, 13, 19$ are obtained.

We should stress once more that our approach is only based on sigma-model quantization methods. It remains to be seen whether a consistent (super)string theory ghost-free spectrum can be built on the entire set of the allowed dimensions (C.14), stemming from our procedure.

Added references

[10] J. Alexandre, J. Ellis and N.E. Mavromatos, *Non-perturbative formulation of time-dependent string solutions*, *JHEP* **12** (2006) 071 [hep-th/0610072].

[11] M.B. Green, J.H. Schwarz and E. Witten, *Superstring theory, vols 1 and 2*, Cambridge Univ. Press, (1987).

[12] I. Antoniadis, C. Bachas, J.R. Ellis and D.V. Nanopoulos, *Comments on cosmological string solutions*, *Phys. Lett.* **B 257** (1991) 278; *An expanding Universe in string theory*, *Nucl. Phys.* **B 328** (1989) 117; *Cosmological string theories and discrete inflation*, *Phys. Lett.* **B 211** (1988) 393.

[13] F. David, *Conformal field theories coupled to 2D gravity in the conformal gauge*, *Mod. Phys. Lett.* **A 3** (1988) 1651;
J. Distler and H. Kawai, *Conformal field theory and 2D quantum gravity or who's afraid of Joseph Liouville?*, *Nucl. Phys.* **B 321** (1989) 509;
see also: N.E. Mavromatos and J.L. Miramontes, *Regularizing the functional integral in 2D quantum gravity*, *Mod. Phys. Lett.* **A 4** (1989) 1847;
E. D'Hoker and P.S. Kurzepa, *2-d quantum gravity and Liouville theory*, *Mod. Phys. Lett.* **A 5** (1990) 1411.

[14] J.-L. Gervais and A. Neveu, *Locality in strong coupling Liouville field theory and string models for seven-dimensions, thirteen-dimensions and nineteen-dimensions*, *Phys. Lett.* **B 151** (1985) 271;
see also: J.-L. Gervais, LPTENS-87-39, Lectures at *Int. Summer School on Conformal Invariance and String Theory*, Poiania Brasov, Romania, Sep 1-12, 1987, Published in *Copenhagen String Th.1987:256 (QCD161:N451:1987)*, and references therein.

JHEP12(2006)071